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On Multiple Parallel Channels with a Finite Source

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Abstract: **In this paper we consider the system of '*l*' parallel channels with a finite source. The queue discipline is first-come-first-served, and the problem is solved by using matrix calculus for the time dependent case. Some useful results in a concise manner have been given. Lastly, the procedure for time dependent case has been illustrated by a simple example.**

Keywords: **Multiple Parallel Channels, system, dependent case, simple example.**

1. INTRODUCTION

Formulation of the problem

We consider the system having l parallel service channels where the units are served according to the negative exponential distribution with mean $1/\mu$. The units arrive independently of each other with calling rate λ per source per unit of the idle time following the Poisson distribution. Earlier the problem for steady-state case was considered by Saaty [2], Jackson and Henderson [1] for infinite number of sources. We assume the number of sources to be finite, say N, and as soon as the units in service depart from the system next units come for service and consequently there is an input in the system depending o the state k of the system (i.e. on the number of busy sources). It will be assumed here that the input λ_k is proportional to the number of idle sources, that is

$$\lambda_k = \lambda(N-k) \quad (k = 0,1,2,\ldots,N).$$

Saaty [2] studied the time dependent solution of the problem taking infinite number of sources using the generating functions and Laplace transform techniques. Later on, Jackson and Henderson [1] studied the same problem but used the slightly modified Laplace transform procedure. Here we shall solve the problem for time dependent case using an alternative method which is quite easy in comparison to the generating functions technique, imposing the condition of finiteness stated above.

We assume the first-come-first-served discipline and also that initially there are i units present in the system.

Equations and their solution

Let $p(k,t)$ denotes the transition probability of the event that at time t the system is in the state k whle initially it is in the state i. The differential-difference equations are:

$$dp(0,t)/dt = -\lambda N p(0,t) + \mu p(1,t) \qquad (1)$$

$$dp(k,t)/dt = \lambda(N-k+1)\,p(k-1,t) - \left[\lambda(N-k)+k\mu\right]p(k,t) + (k+1)\mu p(k+1,t) \quad 0<k<l \qquad (2)$$

$$dp(k,t)/dt = \lambda(N-k+1)\,p(k-1,t) - \left[\lambda(N-k)+l\mu\right]p(k,t)\,l\mu\,p(k+1,t)$$
$$1 \le k < N \qquad (3)$$

$$dp(N,t)/dt = \lambda p(N-1,t) - l\mu p(N,t) \qquad (4)$$

with initial condition $p(k,) = \delta_{ik}$.

1. Steady-State Case. This case has been discussed by many authors (See Saaty [3]). The steady-state equations can be obtained simply by putting

$$\frac{dp(0,t)}{dt}, \frac{dp(k,t)}{dt} \text{ and } \frac{dp}{dt}(N,t)$$

equal to zero in eqns. (1)-(4). The then system of eqns. (1)-(4) becomes on some simplification,

$$(k+1)\mu p(k+1) - (N-k)\lambda p(k) = 0, 0 \le k < l$$

$$l\mu p(k+1) - (N-k)\lambda p(k) = 0, 1 \le k \le N-1 \qquad (5)$$

The solution of eqns. (5) is obtained by simply putting k = 0, 1, 2, 3, …, N – 1 and simplifying the resulting equations, we get

$$p(k) = p(0)\left(\frac{\lambda}{\mu}\right)^k \frac{N!}{k!(N-k)!}, \le k < l$$

$$= p(0)\left(\frac{\lambda}{\mu}\right)^k \frac{N!}{k!(N-k)!} l^{l-k} \frac{k!}{l!}, \; l \le k \le \qquad (6)$$

$p(0)$ can be determined from the condition

$$\sum_{k=0}^{N} p(k) = 1$$

2. Time Dependent Case. We write the above system of equations as (using bold letters for matrix representation):

$$(\theta\mathbf{I} - \mathbf{A})\,\overline{p}(k,t) = 0 \qquad (7)$$

where $\theta \equiv d/dt, \overline{0}$ the null matrix,

$\mathbf{I}$ is the $(N+1)\times(N+1)$ identity matrix

$$\mathbf{A} = \begin{vmatrix} -\lambda N & \mu & 0 & 0 & \cdots & \cdots & & 0 \\ \lambda N & -[\lambda(N-1)+\mu] & 2\mu & 0 & \cdots & \cdots & & 0 \\ 0 & \lambda(-1) & -[\lambda(N-2)+2\mu] & 3\mu & 0 & \cdots & & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & & 0 \\ \vdots & \cdots & \cdots & \cdots & \lambda[N-(l-2)] & -[\lambda\{N-(l-1)\}+(l-1)\mu] & l\mu\ldots & 0 \\ \vdots & \cdots & \cdots & \cdots & \lambda[N-(l-1)] & -[\lambda(N-l)+l\mu] & l\mu\ldots & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & 2\lambda & -[\lambda+l\mu] & l\mu \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda & -l\mu \end{vmatrix}$$

and $$\bar{p}(k,t)=\begin{bmatrix} p(0,t) \\ p(1,t) \\ p(2,t) \\ \vdots \\ p(N,t) \end{bmatrix}$$

Let $\mathbf{C}^{-1}$ and $\mathbf{C}$ be the left and right eigenmatrices of $\mathbf{A}$, then

$$\mathbf{C}^{-1}\mathbf{A}\mathbf{C}=\mathbf{D} \tag{8}$$

where $\mathbf{D}=\mathbf{diag}(d_0,d_1,d_2,\ldots,d_N), d_0,d_1,d_2,\ldots,d_N$ being the eigen-values of $\mathbf{A}$.

premultiplying (7) by $\mathbf{C}^{-1}$ and using (8), we get

$$(\theta\mathbf{I}-\mathbf{D})\mathbf{G}(k,t)=\bar{0} \tag{9}$$

where $\mathbf{G}(k,t)=\mathbf{C}^{-1}\bar{p}(k,t)$ (10)

and $\mathbf{C}$ is the transformation matrix. If we determine $\mathbf{G}(k,t)$ then $\bar{p}(k,t)$ can be determined by the inverse transform

$$\bar{p}(k,t)=\mathbf{C}\,\mathbf{G}(k,t) \tag{11}$$

Eqn. (9) is a Linear Matrix Differential Equation in $\mathbf{G}(k,t)$ whose solution is given by

$$\mathbf{G}(k,t)=\exp(\mathbf{D}t)\mathbf{L} \tag{12}$$

Where $\mathbf{L}=\mathbf{G}(k,0)=\mathbf{C}^{-1}\bar{p}(k,0)$ and $\bar{p}(k,0)$ is a column vector, all of whose elements are 0 except the *i*th, which is 1.

Hence we get

$$\bar{p}(k,t)=\mathbf{C}\exp(\mathbf{D}t)C^{-1}\bar{p}(k,0) \tag{13}$$

3. Some Other Results. If c_{k^r} is the element of matrix C in kth row and rth column and c'_{r_i} is the element in rth row of the resultant multiplication matrix of $\mathbf{C}^{-1}$ and $\bar{p}(k,0)$ (i.e., $\mathbf{C}^{-1}\bar{p}(k,0)$; where $\bar{p}(k,0)$ is the column vector with all elements zero except the *i*th, which is 1), then the calling rate R for the system can be determined from

$$R=\sum_{k=0}^{N}\lambda_k p(k,t)=\sum_{k=0}^{N}\lambda(N-K)C_{k_r}C'_{ri}+\sum_{r,k=0}^{N}C_{kr}C'_{r_i}d_k t\lambda(N-K)$$

$$+\frac{1}{2!}\sum_{r,k=0}^{N}\lambda(N-k)C_{kr}C'_{ri}d_k^2t^2+\cdots+\cdots \text{ time dependent case.}$$

$$=\sum_{k=0}^{N}\lambda_k p(k) \qquad \text{steady-state case} \tag{14}$$

where $\lambda_k=\lambda(N-k),(k=0,1,2,\ldots,N)$ and $p(k)$ is given by (6).

The traffic offered is given by

$$T_r=\frac{R}{-}=\sum_{r,k=0}^{N}\left(\frac{\lambda}{\mu}\right)(N-k)C_{kr}C'_{ri}$$

$$+\sum_{r,k=0}^{N}\frac{\lambda}{\mu}(N-k)C_{kr}C'_{ri}d_k t$$

$$+\frac{1}{2!}\sum_{r,k=0}^{N}\frac{\lambda}{\mu}(N-k)C_{kr}C'_{ri}d_k^2t^2+\ldots$$

Note that the traffic intensity in our case is $l=\lambda/l\mu$. (15)

The termination rate β can be determined from:

$$\beta=\sum_{k=0}^{N}\mu_k p(k,t)=\left[\sum_{r,k=0}^{l_{-1}}k\mu C_{kr}C'_{ri}+\sum_{r,k=0}^{l_{-1}}k\mu C_{kr}C'_{ri}d_k t+\frac{1}{2!}\sum_{r,k=0}^{l_{-1}}k\mu C_{kr}C'_{ri}d_k^2t^2+\ldots+\ldots\right]$$

$$+\left[\sum_{r,k=l}^{N}l\mu C_{kr}C'_{ri}+\sum_{r,k=l}^{N}l\mu C_{kr}C'_{ri}d_k t+\frac{1}{2!}\sum_{r,k=l}^{N}l\mu C_{kr}C'_{ri}d_k^2t^2+\ldots\right] \text{ time-dependent case.}$$

$$=\sum_{k=0}^{N}\mu_k p(k) \qquad \text{steady-state case.} \tag{16}$$

where $\mu_k=k\mu$ for $0\le k\le l-1$ and l_μ for $k\ge l$.

In equilibrium, traffic offered is given by

$$T_r=\frac{R}{\mu}=\frac{\beta}{\mu}=\frac{1}{\mu}\sum_{k=0}^{N}\mu_k p(k)=\frac{1}{\mu}\left[\sum_{k=0}^{l-1}k\mu p(k)+\sum_{k=l}^{N}l\mu p(k)\right]$$

$$=\sum_{k=0}^{l-1}kp(k)+l\sum_{k=l}^{N}p(k) \tag{17}$$

since the termination rate coincides with the calling rate.

Let us define the waiting traffic by

$$T_{rw}=\sum_{k=l}^{N}(k-l)\,p(k,t)=\sum_{r,k=l}^{N}(k-l)C_{kr}C'_{r1}+\sum_{r,k=l}^{N}(k-l)C_{kr}C'_{ri}d_k t$$

$$+\frac{1}{2!}\sum_{r,k=l}^{N}(k-l)C_{kr}C'_{ri}d_k^2t^2\ldots \quad \text{for time dependent case}$$

$$=\sum_{k=l}^{N}(k-l)\,p(k) \quad \text{for steady-state case} \tag{18}$$

then the average number of busy sources or alternatively, the average number of calls either being served or awaiting service is given by

$$E(Y)=T_r+T_{rw} \tag{19}$$

Example. Let us consider the simple case, where $\lambda=1,\mu=1,l=2,N=2$.

We have

$$\mathbf{A}=\begin{bmatrix}-2 & 1 & 0\\ 2 & -2 & 2\\ 0 & 1 & -2\end{bmatrix},\ \bar{p}(k,t)=\begin{bmatrix}p(0,t)\\ p(1,t)\\ p(2,t)\end{bmatrix}$$

The eigenvalues of **A** are $d_0 = 0, d_1 = -2, d_2 = -4$.

$$\mathbf{C} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}, \mathbf{C}^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 2 & 2 \\ -4 & 0 & 4 \\ 2 & -2 & 2 \end{bmatrix}$$

Assuming $i = 0$, we may write $\overline{p}(k,0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and taking $e^{Dt} = I + Dt + \frac{D^2t^2}{2!} + \ldots,$ (where D is the diagonal matrix and t a scalar), we have

$$\overline{p}(k,t) = \begin{bmatrix} p(0,t) \\ p(1,t) \\ p(2,t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\left\{ I + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} t + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}^2 \frac{t^2}{2!} \ldots \right\}$$

$$\frac{1}{8}\begin{bmatrix} 2 & 2 & 2 \\ -4 & 0 & 4 \\ 2 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2t \\ 2t \\ 0 \end{bmatrix} + \begin{bmatrix} 3t^2 \\ -4t^2 \\ t^2 \end{bmatrix} + \ldots$$

$$\therefore p(0,t) = 1 - 2t + 3t^2 + \ldots,$$

$$p(1,t) = 2t - 4t^2 + \ldots,$$

$$p(2,t) = t^2 + \ldots$$

Thus all the probabilities are obtained as polynomials in t.

REFERENCES

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